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NON-PERTURBATIVE ANALYSIS OF THE INFRARED PROPERTIES OF QED_3

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We analyse the Dyson–Schwinger equations of the photon and electron propagator of massless QED_3 Although the perturbation expansion in this theory contains infrared divergences, the Dyson–Schwinger equations admit well-defined solutions Perturbative infrared divergences are replaced, in the solutions of the Dyson–Schwinger equations, by new parameters, which are matrix elements of field operators The full electron propagator is shown to depend crucially on the effective photon mass $\langle A^\mu A_\mu \rangle$ We point out the relevance of our approach to the analysis of Dyson–Schwinger equations in other theories

1. Introduction

Three-dimensional quantum field theories have a very interesting infrared structure, and thus provide us with an excellent testing ground for the study of the long-distance behaviour of quantum field theory in general*. At first sight it appears that massless three-dimensional gauge theories contain insurmountable perturbative infrared divergences. In this paper we shall discuss these non-trivial infrared problems from a non-perturbative point of view. For simplicity's sake we confine ourselves to quantum electrodynamics (QED_3), although much of our analysis is applicable to quantum chromodynamics (QCD_3) as well. These theories are interesting in their own right, as their euclidean versions are the infinite temperature limits of their four-dimensional counterparts. Our purpose is to extract the infrared behaviour of the full photon and electron propagator from an analysis of the Dyson–Schwinger equations of QED_3. We will argue that non-perturbative effects are essential for an understanding of the infrared properties of QED_3. In this respect QED_3 differs radically from QED_4. Nevertheless we stress that such non-perturbative effects in the infrared region may also occur in certain four-dimensional theories. The techniques we present for the analysis of Dyson–Schwinger equations in QED_3 are therefore relevant to those theories as well. In particular, it is an attractive hypothesis that in QCD_4, due to non-perturbative effects, the gluon propagator develops a double pole.

* For an introduction to infrared problems in three dimensions, see for example [1, 2]

In 3+1 dimensions a double pole in the gluon propagator, i e. the behaviour (suppressing all indices)

$$D(p) \sim \frac{A}{p^4} + \text{less singular terms} \tag{1.1}$$

for small p^2, gives rise to a linearly rising potential between two static coloured sources at large distances. Considerable effort has been invested in attempts to show that the Dyson–Schwinger equations actually do have a solution with such a behaviour. In axial gauge this approach was initiated by Baker, Ball and Zachariasen [3], who argued that a particular linearized form of the Dyson–Schwinger equations does have a solution with the behaviour (1.1). This has recently been established rigorously [4], but the simplifications which are required to obtain manageable equations make an interpretation of the result difficult. Mandelstam's approach [5] in covariant gauge, had led to a formal proof [6] that the behaviour (1.1) is consistent with a approximation of the Dyson–Schwinger equations (for a more sophisticated treatment, see [7]). The gluon propagator is a gauge dependent object, so these two approaches need not lead to the same result. In a sense it is surprising that in both gauges the behaviour (1.1) seems to be possible. This obviously raises the question of the nature of the constant A in (1.1), which has the dimension of $(\text{mass})^2$. It is tempting to identify A with the slope of the linear potential, but one should realize that A must be proportional to the matrix element $\langle A^a_\mu(x) A^{a\mu}(x)\rangle$, which is gauge dependent. It is important to understand how such vacuum-expectation values arise from Dyson–Schwinger equations, and to ensure that they are properly defined.

We argue that in massless QED_3 a phenomenon very much like the one sketched for QCD_4 occurs: the infrared behaviour of the theory crucially depends on the gauge dependent effective photon mass $\langle A_\mu A^\mu \rangle$. Besides the unit operator this is the lowest dimensional operator of which the expectation value can enter in the electron propagator. Of course the gauge invariant vacuum polarization should not depend on this effective mass. In our approximation scheme that is indeed true.

It is important to stress that approximations appear to be an unavoidable aspect of the analysis of Dyson–Schwinger equations. The full equations relate n-point functions to $(n+k)$-point functions, and this open set of equations has to be truncated or otherwise short-circuited to obtain a closed, non-linear, system. This truncation usually involves ansätze for $(n+k)$-point functions in terms of n-point functions. For example, in quarkless QCD in covariant gauges one typically sets the 4-point function equal to zero, and makes an ansatz for the 3-point function. In QED an ansatz for the $ee\gamma$ vertex function Γ_μ is required. As the vertex functions must satisfy Ward–Takahashi identities, these ansätze are restricted, but the identities are not sufficiently strong to determine the vertices completely (an exception to the rule is the Schwinger model (QED_2), see below. It is always allowed to add transverse parts to the ansätze for the vertices, which are not constrained by the identities.

In this paper we use a modification of the Salam–Delbourgo gauge technique [8] to make an ansatz for the vertex in QED_3. We find that the usual form of the

Salam–Delbourgo ansatz is inconsistent in QED_3. It causes the appearance of perturbatively infrared divergent, gauge dependent matrix elements in the vacuum polarization. We show that a modification is possible which renders the resulting vacuum polarization manifestly gauge invariant. The consistency of the infrared behaviour of the propagators thus restricts the ansatz for the vertex function, and determines (to the order in which we analyse the equations) the additional transverse contribution to the vertex function. The modified vertex is remarkably similar to the expression which was obtained recently in a similar context [9] in QED_2. There it is the chiral Ward identity which provides additional information, sufficient to permit a complete solution for the vertex function, and thus of the whole model.

QED_3 has been the subject of a number of previous analyses which are relevant to the present work. Jackiw and Templeton [10] argued that massless QED_3 avoids a possible infrared catastrophe, but that amplitudes (in particular the electron propagator) contain logarithms of the coupling constant – typically of the form $e^4 \ln (-p^2)^{1/2}/e^2$ – the coefficients of which are calculable perturbatively. Following 't Hooft's [1] analysis of QCD_3, they stress the presence of contributions which cannot be calculated perturbatively, and are related to matrix elements of composite operators. It was shown later by Templeton [11] that the leading logarithms can actually be obtained to all orders in perturbation theory, and that this series has a Borel sum for appropriate values of e^2. It was stressed by Guendelman and Radulovic [12] that non-analyticity in the coupling constant cannot be established in this way, since the electron propagator is a gauge dependent object. In particular, they showed that the scale of the logarithm, which in [10, 11] was taken to be the coupling constant e, is gauge dependent.

All of these results follow from a careful analysis of the Dyson–Schwinger equations with a proper ansatz for the vertex function. Additional features arise as well. We recover the leading logarithms, but in our approach we can identify the gauge dependent matrix element which sets the scale of the logarithms. We obtain an integral equation for the spectral function of the electron propagator, which can in principle be solved. Here we limit ourselves to an analysis of a slightly simplified equation, in which non-leading infrared effects are neglected. Throughout it is clear which steps are required to go beyond the present stage of the calculation.

It is important to note that inserting a bare photon propagator in the electron Dyson–Schwinger equation is not a good approximation in the infrared region, this in contradistinction to QED_4. This is due to the fact that perturbative infrared divergences first occur in the two-loop electron self-energy due to the correction to the photon propagator. The leading infrared behaviour is therefore determined by the photon propagator, dressed so as to include at least this effect. In our equation for the electron propagator this feature is indeed present.

Massless QED_3 remains massless in perturbation theory because mass terms for the photon and electron violate parity (but not gauge invariance!). Nevertheless, non-perturbative effects may generate masses, as has been discussed recently [13].

Clearly it is important to understand how such effects are incorporated in the Dyson–Schwinger equations. In our approach the electron propagator depends crucially on an effective photon mass, although parity remains unbroken.

This paper is organized as follows. In sect. 2 we discuss the Salam–Delbourgo ansatz in QED_3 and show where it fails. A modification is introduced in sect. 3, and the resulting equations are analyzed in sect. 4. We conclude in sect. 5, and we have added an appendix to establish notation and to explain some properties of spectral functions in QED_3

2. The Salam–Delbourgo ansatz in QED_3

The Dyson–Schwinger equations for the photon and electron propagator involve the unknown vertex functions $\Gamma_\mu(p,p')$. The longitudinal part of Γ_μ is related to the electron propagator S by Ward's identity:

$$k^\mu S(p+k)\Gamma_\mu(p+k,p)S(p)=S(p)-S(p+k)\,. \tag{2.1}$$

The starting point of our analysis of massless QED_3 is an ansatz which respects this identity. Making use of the Källen–Lehmann spectral representation

$$S(p)=\int_{-\infty}^{\infty}\mathrm{d}\omega\,\rho(\omega)\frac{1}{\not p-\omega+i\varepsilon\,\mathrm{sgn}\,\omega}\,, \tag{2.2}$$

Salam and Delbourgo made the following ansatz for Γ_μ [8]:

$$S(p+k)\Gamma_\mu(p+k,p)S(p)=\int_{-\infty}^{\infty}\mathrm{d}\omega\,\rho(\omega)\frac{1}{\not p+\not k-\omega}\gamma_\mu\frac{1}{\not p-\omega}\,, \tag{2.3}$$

where the "$i\varepsilon$ sgn ω" prescription is to be understood in both denominators, and in subsequent formulae. One readily verifies that (2.3) satisfies (2.1), and that the lowest order in perturbation theory is recovered by setting

$$\rho(\omega)=\delta(\omega)\,. \tag{2.4}$$

In the limit of zero photon momentum, k, the ansatz is the exact solution of Ward's identity in differential form:

$$S(p)\Gamma_\mu(p,p)S(p)=-\frac{\partial}{\partial p^\mu}S(p)\,. \tag{2.5}$$

This ansatz was successfully applied in QED_4 [14], where it yields the correct infrared behaviour of the electron propagator. However, there is no reason to expect that (2.3) correctly represents the full vertex function. In principle one can add terms which are transverse and vanish at zero photon momentum on the right-hand side (RHS) of (2.3). This is essential in QED_2, where these transverse contributions are determined by (broken) chiral symmetry [9]. In three dimensions there is no chiral symmetry, so for the moment we shall use (2.3) as it stands. We shall find, however, that in QED$_3$ the requirement of a consistent infrared behaviour forces us to include additional terms in (2.3).

The Dyson–Schwinger equation for the full photon propagator $D_{\mu\nu}$ reads

$$D^{-1}_{\mu\nu}(k) = D^{(0)-1}_{\mu\nu}(k) - \Pi_{\mu\nu}(k)\,, \tag{2.6a}$$

$$\Pi^{\mu\nu}(k) = ie^2 \int \frac{d^3p}{(2\pi)^3} \operatorname{tr}\left[\gamma^\mu S(p+k)\Gamma^\nu(p+k,p)S(p)\right]. \tag{2.6b}$$

One readily calculates the vacuum polarization tensor $\Pi^{\mu\nu}$ by inserting the ansatz (2.3). The result is

$$\Pi^{\mu\nu}(k) = -\frac{e^2}{2\pi}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) \int_{-\infty}^{\infty} d\omega\, \rho(\omega) \int_0^1 d\alpha \frac{\alpha(1-\alpha)k^2}{(\omega^2 - k^2\alpha(1-\alpha))^{1/2}}$$
$$-\frac{e^2}{4\pi} i\varepsilon^{\mu\nu\lambda}k_\lambda \int_{-\infty}^{\infty} d\omega\, \rho(\omega)\omega \int_0^1 d\alpha \frac{1}{(\omega^2 - k^2\alpha(1-\alpha))^{1/2}}. \tag{2.7}$$

The second term in (2.7) gives rise to a contribution to the inverse photon propagator which is of the form

$$D^{-1}_{\mu\nu}(k) \sim i\varepsilon_{\mu\nu\lambda}k^\lambda F(k^2)\,. \tag{2.8}$$

Such a term is also present in the propagator of a massive photon in QED$_3$, as the gauge invariant mass term is proportional to the Chern-Simons term $\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho$ [15]. Thus we see from (2.7) that a mass term for the electron, i.e. $\rho(\omega) = \delta(\omega - m)$ in lowest order, generates a non-zero photon mass. As we are primarily interested in the massless theory we want to avoid having (2.8) with $F(k^2) \neq 0$. This can be achieved by requiring $\rho(\omega)$ to be an even function of ω. As we shall see, this requirement is consistent with the Dyson–Schwinger equations.

As is well known [15] the electron mass term and the gauge invariant photon mass term in three dimensions violate parity. In the appendix we show that the conservation of parity indeed requires the electron spectral function $\rho(\omega)$ to be an even function of ω. Our choice of an even function $\rho(\omega)$ thus sets parity-violating expectation values equal to zero, e.g.

$$\langle \psi\bar{\psi} \rangle = \frac{1}{\pi}\int_0^\infty d\omega\, \omega^2(\rho(\omega) - \rho(-\omega)) = 0\,. \tag{2.9}$$

Let us now proceed with the analysis of (2.7). After performing the α-integration, one finds for the imaginary part of $\Pi^{\mu\nu}$:

$$\operatorname{Im} \Pi^{\mu\nu}(k) = -\Delta^{\mu\nu}(k)\frac{e^2}{8k}\int_0^{k/2} d\omega\, \rho(\omega)(k^2 + 4\omega^2)\,, \tag{2.10}$$

where

$$\Delta^{\mu\nu}(k) = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}. \tag{2.11}$$

To lowest order, $\rho(\omega)=\delta(\omega)$, one finds

$$\Pi^{\mu\nu}(k)=+\Delta^{\mu\nu}\tfrac{1}{16}e^2\sqrt{-k^2}\,, \tag{2.12}$$

$$D^{\mu\nu}(k)=-\Delta^{\mu\nu}\frac{1}{k^2-\frac{1}{16}e^2\sqrt{-k^2}}-a\frac{k^\mu k^\nu}{k^4}\,. \tag{2.13}$$

It is convenient to introduce a spectral function for the photon propagator as well. We write (in arbitrary covariant gauge)

$$D^{\mu\nu}(k)=-\Delta^{\mu\nu}(k)\int_0^\infty \mathrm{d}\mu\,\frac{\bar\rho(\mu)}{k^2-\mu^2+i\varepsilon}-a\frac{k^\mu k^\nu}{k^4}\,, \tag{2.14}$$

and find to this order

$$\bar\rho(\mu)=\frac{e^2}{8\pi}\frac{1}{\mu^2+(\frac{1}{16}e^2)^2}\,. \tag{2.15}$$

This result, previously obtained in [16] and [10], is to be trusted only for $e^2/\mu \ll 1$.

The expression (2.10) reveals the presence of perturbative infrared divergences in QED$_3$. On general grounds, one expects that the perturbative electron propagator will be a series in e^2/ω, possibly modified by logarithms. Therefore, one expects infrared divergences in the perturbative expansion of Im $\Pi_{\mu\nu}$ at the lower limit of the ω-integration. The way to proceed in a perturbative analysis of Dyson–Schwinger equations is to replace such infrared divergent integrals by

$$\int_0^{k/2}\mathrm{d}\omega\,\omega^n\rho(\omega)=\int_0^\infty \mathrm{d}\omega\,\omega^n\rho(\omega)-\int_{k/2}^\infty \mathrm{d}\omega\,\omega^n\rho(\omega)\,, \tag{2.16}$$

and to relate ω-integrals extending from zero to infinity to expectation values of field operators. In our case that gives

$$\mathrm{Im}\,\Pi^{\mu\nu}(k)=-\Delta^{\mu\nu}(k)\tfrac{1}{16}ke^2\left[1+\frac{8A}{k^2}-2\int_{k/2}^\infty \mathrm{d}\omega\,\rho(\omega)\left[1+\frac{4\omega^2}{k^2}\right]\right]. \tag{2.17}$$

Here we have used the Lehmann sum rule

$$\int_{-\infty}^\infty \mathrm{d}\omega\,\rho(\omega)=1\,, \tag{2.18}$$

and we have introduced

$$A=\left.\int_0^\infty \mathrm{d}\omega\,\omega^2\rho(\omega)\right|_{\text{UV regulated}}\,, \tag{2.19}$$

which is related to the matrix element $\langle\bar\psi(\not\partial\psi)\rangle$ (see appendix). However, A is not gauge invariant, and should not occur in the vacuum polarization. Therefore, the Salam–Delbourgo ansatz is not consistent in massless QED$_3$.

Before introducing an improved ansatz in the next section, we briefly study the consequences of (2.3) for the electron propagator. The Dyson–Schwinger equation reads

$$\not{p}S(p)=\mathbb{1}+\Sigma(p)S(p)\,, \tag{2.20a}$$

$$\Sigma(p)S(p)=\frac{ie}{(2\pi)^3}\int d^3k\,\gamma_\mu S(p+k)\Gamma_\nu(p+k,p)S(p)D^{\mu\nu}(-k)\,. \tag{2.20b}$$

Inserting a bare photon in the electron self-energy, and substituting (2.3), we find to lowest order in e^2/p

$$p\rho(p)=a\frac{e^2}{p}\,, \tag{2.21}$$

where a is the gauge parameter. The resulting Im $\Pi_{\mu\nu}$ (eq. (2.10)) can be calculated using (2.16). This involves only the sum rule (2.18), and one finds that the a-dependence cancels. This is of course consistent with gauge invariance, and shows that the ansatz (2.3) is acceptable to order e^4 in QED_3. The appearance of the gauge dependent matrix element A (2.19), and thus the inconsistency of the ansatz, first occur at order e^6 in the vacuum polarization

The electron spectral function to order e^4 can easily be extracted from (2.20), together with (2.3). Inserting $\rho(\omega)=\delta(\omega)$ in the RHS of (2.20b) we find, after taking imaginary parts (we now limit ourselves to the Landau gauge ($a=0$)):

$$\rho(p)=-\frac{e^2}{16\pi}\left\{\int_p^\infty d\mu\,\bar{\rho}(\mu)\frac{1}{\mu^2}+\int_0^p d\mu\,\bar{\rho}(\mu)\frac{\mu^2}{p^4}\right\}. \tag{2.22}$$

Expanding $\bar{\rho}(\mu)$ (2.15) to lowest order in e^2, we find

$$\rho(p)=-\frac{e^4}{96\pi^2p^3}+O(e^6)\,. \tag{2.23}$$

This result allows us to reconstruct the electron propagator to order e^4:

$$\not{p}S(p)=\int_{-\infty}^{\infty}d\omega\,\rho(\omega)\frac{p^2}{p^2-\omega^2}=1+\frac{2}{p^2}A(\Lambda)-\frac{e^4}{48\pi^2p^2}\ln\frac{(-p^2)^{1/2}}{\Lambda}\,, \tag{2.24}$$

where $A(\Lambda)$ is given by (2.19), with an ultraviolet cut-off Λ. Thus the electron propagator develops a logarithm at the order e^4 [10]. Its coefficient is calculable in perturbation theory, and fixed by the coefficient of the e^4/p^3 term in the spectral function $\rho(p)$. The scale of the logarithm is not calculable in perturbation theory, but is related to the gauge dependent matrix element $A(\Lambda)$.

3. A modified ansatz

In this section we shall construct an ansatz which cures the deficiencies of the Salam–Delbourgo Ansatz we signalled in the previous section. These deficiencies are related to the appearance of the gauge dependent, perturbatively infrared-divergent matrix element A in the vacuum polarization (2.17) and the electron propagator (2.24)

The first observation is that in covariant gauges such as we employ here, a certain amount of gauge freedom remains. In particular, one still has the freedom to perform constant gauge transformations

$$A_\mu(x) \to A_\mu(x) + \frac{g_0}{e} B_\mu , \tag{3.1}$$

where B_μ is a constant vector. In massless QED$_3$ such gauge transformations were studied in [12]. In particular, it was shown that the scale of the logarithm which appears in the electron propagator (2.24) can be given an arbitrary value by these constant gauge transformations. This suggests that the appearance of the constant A in (2.24), which is related to this scale, can be avoided by a suitable B-gauge transformation. However, in the context of the ansatz (2.3), the constant A cannot be completely a gauge artefact, since it appears in the vacuum polarization. As we shall show, it is possible to modify (2.3) in such a way that in the new ansatz the constant A *does* become a gauge artefact, and will therefore be automatically absent from $\Pi_{\mu\nu}$.

Let us note a few consequences of the gauge freedom (3.1). In the first place, in the context of the Dyson–Schwinger equation (2.6), it is the inverse photon propagator which is actually calculated. The determination of the propagator itself then contains an ambiguity if $\Pi(0)=0$: at zero momentum the propagator is ill-defined by (2.6). This possibility of modifying the photon propagator by contributions from zero-momentum "B-photons" is precisely reflected in the gauge freedom (3.1).

Under (3.1) the electron propagator equation (2.20) is changed to

$$(\not p + g_0 \not B) S(p) = \mathbb{1} + \Sigma(p) S(p) . \tag{3.2}$$

Of course the vacuum polarization should not change. In perturbation theory to lowest order, this is easily seen. There we find with (3.2):

$$\Pi^{\mu\nu}(k) = \frac{ie^2}{(2\pi)^3} \int \mathrm{d}^3 p \, \mathrm{tr} \left[\gamma^\mu \frac{1}{\not p + \not k + g_0 \not B} \gamma^\nu \frac{1}{\not p + g_0 \not B} \right] . \tag{3.3}$$

After the change of integration variable $p \to p + g_0 B$ the answer is manifestly independent of B.

One may supplement the generating functional with the A_μ field shifted according to (3.1) with a gauge-fixing term. In [12] QED$_3$ was studied in perturbation theory with the gauge fixing term $-\frac{1}{2}B^2$. In the context of Dyson–Schwinger equations this amounts to replacing (3.2) by

$$S(p) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 B \frac{\mathrm{e}^{-B^2/2}}{\not p + g_0 \not B} (\mathbb{1} + \Sigma(p) S(p)) . \tag{3.4}$$

On expanding (3.4) to order e^4 and g_0^2, one finds

$$S(p)=\frac{\not p}{p^2}\left[1-\frac{g_0^2}{p^2}+2\frac{A(\Lambda)}{p^2}-\frac{e^4}{48\pi^2 p^2}\ln\frac{(-p^2)^{1/2}}{\Lambda}\right], \tag{3.5}$$

showing that indeed the scale of the logarithm is affected by the constant gauge transformations, and that the constant A can be cancelled by a suitable B-photon counterterm. The procedure of incorporating the B-photons non-perturbatively through (3.4) is reminiscent of the results of [11], where an effective theory for the leading and subleading logarithms in massless QED$_3$ was constructed. There, however, g_0 is momentum dependent, invalidating a shift of variable such as in (3.3).

Let us now calculate the modified ansatz. First we determine how the constant A appears in the ansatz (2.3). On expanding the denominators on the RHS of (2.3) in ω we find that A, the integral (2.19), occurs in the combination

$$2A\left[\frac{1}{(p+k)^2}\gamma_\mu\frac{1}{p^2}+\left(\frac{1}{(p+k)^2}+\frac{1}{p^2}\right)\frac{1}{\not p+\not k}\gamma_\mu\frac{1}{\not p}\right] \tag{3.6}$$

The LHS of (2.3) is gauge dependent, and will be modified by B-photon contributions after a gauge transformation (3.1). These contributions, to all orders in g_0, can be conveniently represented by the integral

$$\frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3B\,\mathrm{e}^{-B^2/2}\frac{1}{\not p+\not k+g_0\not B}\gamma_\mu\frac{1}{\not p+g_0\not B}. \tag{3.7}$$

To order g_0^2, this gives

$$\frac{g_0^2}{\not p+\not k}\left[\gamma^\lambda\frac{1}{\not p+\not k}\gamma_\mu\frac{1}{\not p}\gamma_\lambda-\left(\frac{1}{(p+k)^2}+\frac{1}{p^2}\right)\gamma_\mu\right]\frac{1}{\not p}. \tag{3.8}$$

Now note the identity

$$\gamma^\lambda\frac{1}{\not p+\not k}\gamma_\mu\frac{1}{\not p}\gamma_\lambda=-\frac{1}{\not p+\not k}\gamma_\mu\frac{1}{\not p}+4i\varepsilon_{\mu\nu\lambda}\frac{k^\nu p^\lambda}{(p+k)^2p^2}. \tag{3.9}$$

So we see that (3.6) has the form of a B-photon contribution, except that the second term on the RHS of (3.9) is absent. But this term is clearly transverse, and the ansatz (2.3) can be modified, respecting the Ward identity, such that (3.6) and (3.8) do have exactly the same form. With this modification, the presence of the matrix element A becomes purely a gauge artefact. The modified ansatz is

$$S(p+k)\Gamma_\mu(p+k,p)S(p)=\int_{-\infty}^{\infty}\mathrm{d}\omega\,\rho(\omega)\frac{1}{\not p+\not k-\omega}\times\left[\gamma_\mu-4i\varepsilon_{\mu\lambda\rho}k^\lambda p^\rho\frac{\omega^2}{((p+k)^2-\omega^2)(p^2-\omega^2)}\right]\frac{1}{\not p-\omega}. \tag{3.10}$$

Strictly speaking, our argument determines the new term in (3.10) only to order ω^2 (near $\omega = 0$). The additional factors of ω^2 in the denominator ensure that the new term does not introduce ultraviolet divergences. Also, as we shall see in the next section, the form (3.10) leads to particularly simple equations.

As was the case with (2.3), there is no reason to expect that (3.10) correctly represents the full vertex function. At higher orders it is not excluded, and even likely, that other, gauge dependent, perturbatively infrared-divergent matrix elements reappear in the vacuum polarization. In principle we could then repeat the analysis of this section and obtain another, more suitable version of (3.10). There is unfortunately no procedure to determine the correct modification to all orders in perturbation theory. We shall therefore be content with (3.10), and analyze its consequences for the Dyson–Schwinger equations. This point of view is vindicated by the results of the next section, where we show that the Dyson–Schwinger equations supplemented with (3.10) describe the leading infrared structure of QED_3.

Finally, we calculate the vacuum polarization using (3.10), and thus verify that the modification works as it should. We obtain

$$\Pi_{\mu\nu}(k) = -\frac{e^2}{8\pi}\Delta_{\mu\nu}\int_0^\infty d\omega\,\rho(\omega)\left[-4\omega + k\ln\frac{2\omega+k}{2\omega-k} + 16\frac{\omega^3}{4\omega^2-k^2}\right]. \quad (3.11)$$

The imaginary part of (3.11) is

$$\operatorname{Im}\Pi_{\mu\nu}(k) = -\tfrac{1}{8}e^2\left[k\int_0^{k/2} d\omega\,\rho(\omega) + \tfrac{1}{2}k^2\rho(\tfrac{1}{2}k)\right]\Delta_{\mu\nu}. \quad (3.12)$$

Rewriting this using (2.16), we find that this time only the unit operator, through the sum rule (2.18), is involved. The result is

$$\operatorname{Im}\Pi_{\mu\nu}(k) = -\tfrac{1}{16}e^2k\left[1 - 2\int_{k/2}^\infty d\omega\,\rho(\omega) + k\rho(\tfrac{1}{2}k)\right]\Delta_{\mu\nu}, \quad (3.13)$$

which is suited for a perturbative analysis. To lowest order, the electron spectral functions is still given by (2.21). Again, this gives a vanishing contribution to $\operatorname{Im}\Pi_{\mu\nu}$.

4. Analysis of the Dyson-Schwinger equations

Now that we modified the ansatz (2.3) so as to avoid the problems discussed above, it is worthwhile to examine the Dyson–Schwinger equations in more detail, and to go beyond the precursory investigation of sect. 2. At the end of sect. 3 we have already shown that the vacuum polarization is now free of the problems which occurred at order e^6 in perturbation theory, so we shall start here by reconsidering the electron propagator equation (2.20). Using again the spectral representations

(2.2) and (2.14) we obtain with (3.10)

$$\int_{-\infty}^{\infty} \mathrm{d}\omega\, \rho(\omega) \frac{p^2}{p^2-\omega^2} = 1 - \frac{e^2}{2\pi} \int_0^{\infty} \mathrm{d}\omega\, \rho(\omega) \int_0^{\infty} \mathrm{d}\mu\, \bar{\rho}(\mu) K(p, \omega, \mu)$$
$$- \frac{e^2 a}{8\pi} \int_0^{\infty} \mathrm{d}\omega\, \rho(\omega) L(p, \omega)\,, \tag{4.1}$$

where the kernels K and L are given by

$$K(p, \omega, \mu) = \frac{1}{2\mu^2}\left\{ -\mu + \frac{p^2-\omega^2}{2p}\left(\ln\left(1+\frac{\mu}{p+\omega}\right) - \ln\left(1-\frac{\mu}{p-\omega}\right)\right)\right\}$$
$$+ \frac{1}{2}\left(\frac{4\omega^2}{(p^2-\omega^2)^2} - \frac{1}{p^2-\omega^2}\right)\left(\omega - \mu + \frac{1}{2}\frac{\mu^2}{p} \ln \frac{\omega+\mu+p}{\omega+\mu-p}\right), \tag{4.2}$$

$$L(p, \omega) = \frac{1}{p} \ln \frac{\omega+p}{\omega-p} - \frac{2\omega}{p^2-\omega^2}\,. \tag{4.3}$$

Eq. (4.1), and the photon propagator equation, which reads

$$\int_0^{\infty} \mathrm{d}\mu \frac{\bar{\rho}(\mu)}{k^2-\mu^2} = \frac{1}{k^2 - \Pi(k^2)}, \tag{4.4}$$

where we have defined

$$\Pi_{\mu\nu}(k) = \Delta_{\mu\nu}(k) \Pi(k^2)\,, \tag{4.5}$$

constitute a system of coupled integral equations for the unknown functions $\rho(\omega)$ and $\bar{\rho}(\mu)$. The function $\Pi(k^2)$ can be read off from (3.11).

Clearly the system of non-linear equations (4.1)–(4.5) is complicated, and we do not expect to find its general solution. As we are primarily interested in the leading infrared behaviour, we can perform a less ambitious analysis of the equations. In QED_4 the leading infrared behaviour of the electron propagator was found by considering the Dyson–Schwinger equation with a bare photon [14]. In our case such a treatment is insufficient, as the infrared structure of $S(p)$ depends crucially on the dressing of the photon. We shall consider the implications of (4.1) using the expansion of the photon propagator in powers of e^2.

To lowest order, the photon spectral function is still given by (2.15), as can be seen by substituting $\rho(\omega) = \delta(\omega)$ in (3.12). The result (2.15) can be trusted for $e^2/\mu \ll 1$, and we can therefore use its asymptotic form,

$$\bar{\rho}(\mu) = \frac{e^2}{8\pi\mu^2}, \tag{4.6}$$

for $e^2/\mu < 1$ in (4.1). For $e^2/\mu > 1$ we will still use (4.6), except where this causes infrared divergences in the μ integrations. Two types of integrals then remain in

(4.1) that give rise to perturbative infrared divergences:

$$I_n = \int_0^\infty d\mu \, \mu^n \bar{\rho}(\mu) , \qquad n=0,1 . \tag{4.7}$$

For $n=0$ we use again Lehmann's sum rule $I_0=1$ (note that (2.15) satisfies this sum rule). The integral I_1 however remains as an undetermined parameter in the theory: it can only be determined if the full photon propagator is completely known in the infrared. Consequently we introduce the dimensionless constant

$$C(\Lambda) = \frac{8\pi}{e^2} \int_0^\Lambda d\mu \, \mu \bar{\rho}(\mu) , \tag{4.8}$$

where Λ is an unltraviolet cutoff (the integral (4.8) is logarithmically divergent in the ultraviolet). The expression for the electron propagator will thus contain the unknown $C(\Lambda)$.

Following the procedure outlined above, the integration over μ in the electron propagator equation can be evaluated. We find

$$\begin{aligned} \int d\omega \, \rho(\omega) \frac{p^2}{p^2-\omega^2} = 1 - \frac{e^2}{\pi} \int_0^\infty d\omega \, \rho(\omega) \Bigg[& \frac{\omega^3}{(p^2-\omega^2)^2} \\ & + \frac{1}{3} \frac{e^2}{8\pi} \frac{p^2-3\omega^2}{(p^2-\omega^2)^2} \left\{ C(\Lambda) + \ln \frac{(-p^2)^{1/2}}{\Lambda} \right\} + \frac{e^2}{8\pi} R(\omega,p) \Bigg] \\ & - \frac{e^2 a}{8\pi} \int_0^\infty d\omega \, \rho(\omega) L(p,\omega) . \end{aligned} \tag{4.9}$$

The function $R(\omega,p)$ is given by

$$R(\omega,p) = \frac{1}{6} \frac{p^2-3\omega^2}{(p^2-\omega^2)^2} \ln\left(1-\frac{\omega^2}{p^2}\right) - \frac{2}{9} \frac{p^2-6\omega^2}{(p^2-\omega^2)^2} - \frac{2}{3} \frac{\omega^3}{p(p^2-\omega^2)^2} \ln \frac{\omega+p}{\omega-p} . \tag{4.10}$$

Notice that the scale of the logarithm on the RHS is related to the constant $C(\Lambda)$ through the cutoff Λ. If we define

$$\ln \frac{C}{\Lambda} = -\frac{8\pi}{e^2} \int_0^\Lambda d\mu \, \mu \bar{\rho}(\mu) , \tag{4.11}$$

then, with the asymptotic behaviour (4.6), C is in fact cut-off independent. The logarithm in (4.9) then appears in the form

$$\ln (-p^2)^{1/2}/C , \tag{4.12}$$

and we see that its scale is set by the perturbatively undetermined constant C. From our knowledge of the spectral function we know that

$$\frac{1}{\pi} \int_0^\infty d\mu \, \mu \bar{\rho}(\mu) \Big|_{\text{UV regulated}} = -\langle A_\mu A^\mu \rangle , \tag{4.13}$$

so that the solution for the electron propagator will depend on the gauge dependent expectation value $\langle A_\mu A^\mu \rangle$. Since the electron propagator is gauge dependent this does not come as a surprise. In sect. 3 we already showed that constant gauge transformations (3.1) change the scale of the logarithm. Clearly these transformations change the matrix element (4.13) as well.

To obtain the leading infrared behaviour of the electron propagator we approximate the integrand in (4.9) for small ω. We see that the integrand is dominated by the logarithmic term, and the equation takes on the form (we now limit ourselves to Landau gauge, $a=0$):

$$\int_{-\infty}^{\infty} \mathrm{d}\omega\, \rho(\omega) \frac{p^2}{p^2-\omega^2} = 1 - \frac{e^4}{48\pi^2} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \rho(\omega) \frac{p^2-3\omega^2}{(p^2-\omega^2)^2} \ln \frac{(-p^2)^{1/2}}{C} . \tag{4.14}$$

The leading contribution from $R(\omega, p)$ (see (4.10)) can at most cause a shift in the constant C. It is interesting to note that the contributions on the RHS of (4.14) are all due to the fact that the photon is dressed. Taking the photon bare, i.e. $\bar{\rho}(\mu) = \delta(\mu)$, in (4.1), leaves only the first term in (4.9). This can be seen from a small μ-expansion of the kernel K in (4.2):

$$K(p, \omega, \mu) \to \frac{2\omega^3}{(p^2-\omega^2)^2} + \mu \frac{4p^2 - 12\omega^2}{6(p^2-\omega^2)^2} + \mathrm{O}(\mu^2) . \tag{4.15}$$

Thus the dressing of the photon is crucial in obtaining the correct electron propagator behaviour in the infrared.

Eq. (4.14) can be rewritten as a differential equation for the electron propagator. Writing

$$\not{p} S(p) = p^2 f(p^2) , \tag{4.16}$$

(4.14) takes on the form

$$x f(x) = 1 - \lambda (x f'(x) + \tfrac{3}{2} f(x)) \ln \frac{(-x)^{1/2}}{C} , \tag{4.17}$$

where we have introduced

$$x = p^2 , \qquad \lambda = \frac{e^4}{24\pi^2} . \tag{4.18}$$

To get some feeling for the content of this equation, we first substitute the free propagator, corresponding to $f(x) = 1/x$, in the RHS of (4.17). The result is

$$x f(x) = 1 - \tfrac{1}{2}\lambda \frac{1}{x} \ln \frac{(-x)^{1/2}}{C} , \tag{4.19}$$

corresponding to an electron propagator (see (4.16))

$$S(p) = \frac{1}{\not{p}} \left[1 - \tfrac{1}{2}\lambda \frac{1}{p^2} \ln \frac{(-p^2)^{1/2}}{C} \right] . \tag{4.20}$$

A comparison with (2.24) shows that the coefficient of the logarithm is the same, but the gauge-dependent scale is now set by the constant C (4.11), and thus by the matrix element (4.13).

To higher orders in perturbation theory the electron propagator will develop higher powers of $\ln[(-x)^{1/2}/C]$. The leading logarithms, i.e. the contributions of the form

$$c_n\left(\frac{\lambda}{x}\ln\frac{(-x)^{1/2}}{C}\right)^n, \tag{4.21}$$

can be completely determined from (4.17). Substituting the expansion (4.21) into (4.17), and equating leading terms, gives

$$c_n=\frac{\Gamma(n-\frac{1}{2})}{\Gamma(-\frac{1}{2})}. \tag{4.22}$$

Therefore the contribution of these leading logarithms to the electron propagator is

$$S_{\mathrm{LL}}(p)=\frac{1}{\not p}\sum_{n=0}^{\infty}\frac{\Gamma(n-\frac{1}{2})}{\Gamma(-\frac{1}{2})}\left(\frac{e^4}{24\pi^2p^2}\ln\frac{(-p^2)^{1/2}}{C}\right)^n. \tag{4.23}$$

A similar result was obtained by Templeton [11], by summation of the series of leading logarithms as they appear in perturbation theory. In our approach, which is essentially non-perturbative, one finds that the scale of the logarithm is set by the non-perturbative effective photon mass (4.13).

It is interesting to remark that our ansatz (3.10), which as we discussed in sect. 3, may not be complete, is nevertheless sufficiently sensitive to the infrared behaviour to reveal the result (4.23). This is because further modifications will necessarily contain higher powers of ω (at least ω^4 in the numerator), and are therefore less significant for the infrared (small ω) behaviour of the integrand (4.9).

For $p^2<0$, $(e^4/24\pi^2)\ln[(-p^2)^{1/2}/C]>0$, expression (4.23) has Borel sum:

$$S_{\mathrm{LL}}(p)=\frac{1}{\not p}\left[1+\tfrac{1}{2}\sqrt{\pi}g(p)^{1/2}\exp\left(\frac{1}{g(p)}\right)\operatorname{erfc}\left(g(p)^{-1/2}\right)\right], \tag{4.24}$$

where

$$g(p)=\frac{e^4}{24\pi^2p^2}\ln\frac{(-p^2)^{1/2}}{C}. \tag{4.25}$$

In order to elucidate the case $g(p)>0$, and to investigate the possible existence of additional non-perturbative effects, we now discuss the full solution of (4.17). As this is a linear equation of first order, it is easy to integrate. The result is

$$xf(x)=1-\exp(-h(x))\int_{\alpha}^{x}\mathrm{d}y\frac{1}{2y}\exp(+h(y)), \tag{4.26}$$

where α is a constant to be determined from the boundary condition, and

$$h(x)=\int_{x_0}^{x}\mathrm{d}y\left\{\frac{1}{\lambda\ln[(-y)^{1/2}/C]}+\frac{1}{2y}\right\}. \tag{4.27}$$

The lower limit x_0 in (4.27) can remain unspecified, $f(x)$ is independent of its choice. The asymptotic behaviour of $h(x)$ is

$$\operatorname{Re}h(x)=\frac{x}{\operatorname{Re}\lambda}\frac{1}{\ln[|x|^{1/2}/C]}\left\{1+\mathrm{O}\left(\frac{1}{\ln[|x|^{1/2}/C]}\right)\right\} \tag{4.28}$$

for $|x|\to\infty$, and so for $\lambda>0$ $\exp(-h(x))$ will diverge for $x\to-\infty$. To ensure that in the ultraviolet region the solution for the full propagator tends to the bare one, and to avoid exponential ultraviolet divergences, we impose the boundary condition

$$xf(x)\to1 \quad \text{for} \quad |x|\to\infty. \tag{4.29}$$

This yields the solution

$$xf(x)=1-\exp(-h(x))\int_{-\infty}^{x}\mathrm{d}y\frac{1}{2y}\exp(h(y)). \tag{4.30}$$

An asymptotic expansion of this solution reproduces the leading logarithms as given by (4 21) and (4.22). Non-leading corrections are of the form

$$c_{nm}\left(\frac{\lambda}{x}\right)^n\left(\ln\frac{(-x)^{1/2}}{C}\right)^m, \qquad n>m>0. \tag{4.31}$$

Note that (4.30) cannot be naively extrapolated to $\lambda<0$, as in that case $f(x)$ does not satisfy (4.29). For negative values of λ there is nevertheless a solution to (4.17) satisfying (4.29), which differs from (4.30) by a homogeneous solution of (4.17).

The electron propagator corresponding to (4.30) is

$$S(p)=\frac{1}{\not p}\left[1-\int_{-\infty}^{p^2}\mathrm{d}y\frac{1}{2y}\exp\left(\int_{p^2}^{y}\mathrm{d}z\left\{\frac{24\pi^2}{e^4\ln[(-z)^{1/2}/C]}+\frac{1}{2z}\right\}\right)\right]. \tag{4.32}$$

This concludes the present discussion of the electron propagator. So far we have considered the expansion of the photon spectral function to order e^2/μ^2 only. To obtain information about the appearance of further matrix elements, we require the next order contribution to the vacuum polarization.

The photon spectral function is determined by the imaginary part of (4.4):

$$\bar{\rho}(k)=-\frac{2k}{\pi}\operatorname{Im}\frac{1}{k^2-\Pi(k^2)}, \tag{4.33}$$

where $\Pi(k^2)$ is given by (4.5) and (3.11). In lowest order the vacuum polarization is purely imaginary (see (2.12)), and there is no e^4 contribution, as we discussed at the end of sect. 3. Therefore the next term is of order e^6, and is determined by the expansion of the electron propagator, (4.20) or (2.23). We obtain

$$\operatorname{Im} \Pi(k) = -\tfrac{1}{16}e^2 k\left(1 - \frac{e^4}{24\pi^2}\frac{1}{k^2}\right),$$

corresponding to

$$\bar{\rho}(\mu) = \frac{e^2}{8\pi}\frac{1}{\mu^2 + (\frac{1}{16}e^2)^2}\left(1 - \frac{e^4}{24\pi^2\mu^2}\right). \tag{4.34}$$

The crucial point is that this $\bar{\rho}(\mu)$, expanded to order e^6/μ^4, has to be reinserted into (4.1). There the same procedure which we explained at the beginning of this section should be applied, now including the new term. Infrared divergences which arise from the e^6/μ^4 contribution then force us to introduce additional non-perturbative constants in the electron propagator. These will be related to matrix elements of operators of dimensions 2 and 3. At this stage, gauge invariant and parity conserving operators may appear. For instance, the integral

$$I_3 = \int_0^\infty \mathrm{d}\mu\, \mu^3 \bar{\rho}(\mu), \tag{4.35}$$

which has a logarithmic perturbative infrared divergence, is related to the matrix element $\langle F_{\mu\nu}F^{\mu\nu}\rangle$. Therefore the constant I_3 does not need to cancel in the calculation of the vacuum polarization at order e^{10}. We will not work out the details of this calculation, since a proper treatment of other effects at order e^{10} presumably requires a further modification of the ansatz (3.10). We merely point out that a consistent treatment of infrared divergences provides a mechanism by which the dependence of gauge invariant amplitudes on gauge invariant matrix elements can be explicitly analyzed.

5. Conclusions

In general, there is little hope of exactly solving the Dyson–Schwinger equations of a realistic quantum field theory. In this paper we have developed techniques which keep the equation manageable, but leave the physical content of the theory intact. In QED_3, the ansatz for the vertex function was seen to be constrained severely by gauge invariance. Non-perturbative effects, essential for a proper understanding of the theory, occurred through the dependence of the full propagators on new parameters, which are related to composite operators. These techniques should be useful for studying the infrared properties of other theories as well. Thus, in QCD_4, the infrared properties of the gluon Dyson–Schwinger equation need not depend on the matrix element $\langle A^a_\mu A^{\mu a}\rangle$ alone. In particular, one should like to see $\langle F_{\mu\nu}{}^a F^{\mu\nu a}\rangle$ and other gauge invariants emerge as parameters in the full gluon propagator. Evidently then it is necessary to extend the "B-photon" construction we presented in sect. 3 to non-abelian theories.

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Appendix

SPECTRAL FUNCTIONS IN THREE DIMENSIONS

In 3 dimensions the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{A.1}$$

can be realized by 2×2 matrices. A convenient basis is

$$\gamma^0 = \sigma_3\,, \quad \gamma^1 = i\sigma_1, \quad \gamma^2 = i\sigma_2\,. \tag{A.2}$$

Useful relations are

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\varepsilon^{\mu\nu\rho}\gamma_\rho\,, \tag{A.3}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho = g^{\mu\nu}\gamma^\rho + g^{\nu\rho}\gamma^\mu - g^{\mu\rho}\gamma^\nu - i\varepsilon^{\mu\nu\rho}\mathbb{1}\,. \tag{A.4}$$

We work throughout with a single complex two-component spinor $\psi(x, t)$. The operators $\mathscr{P}$, $\mathscr{C}$ and $\mathscr{T}$ of the discrete parity, charge-conjugation and time-reversal transformations act on the field operator $\psi(x, t)$ as follows:

$$\mathscr{P}\psi_\alpha(x, t)\mathscr{P}^{-1} = (i\gamma^1)_{\alpha\beta}\psi_\beta(x', t)\,, \tag{A.5a}$$

$$\mathscr{C}\psi_\alpha(x, t)\mathscr{C}^{-1} = (\gamma^2)_{\alpha\beta}\bar{\psi}_\beta(x, t)\,, \tag{A.5b}$$

$$\mathscr{T}\psi_\alpha(x, t)\mathscr{T}^{-1} = (i\gamma^2)_{\alpha\beta}\psi_\beta(x, -t)\,. \tag{A.5c}$$

Note that in (A.5a) we have $x'_\mu = (-x_1, x_2)$, since in an even number of space dimensions complete reversal corresponds to a rotation. The discrete transformations (A.5) are symmetries of the massless Dirac theory in $d=3$. It is easy to see that a mass term, $m\bar{\psi}\psi$, changes sign under parity and time reversal.

The electron propagator in coordinate space,

$$S_{\alpha\beta}(x-y) = -i\langle 0|T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\}|0\rangle\,, \tag{A.6}$$

can be written in the form

$$\begin{aligned} S_{\alpha\beta}(x) = -i &\int \mathrm{d}^3q\, \theta(x^0)\, \mathrm{e}^{-iqx}\rho_{\alpha\beta}(q)\theta(q^0) \\ &+ i\int \mathrm{d}^3q\, \theta(-x^0)\, \mathrm{e}^{iqx}(\gamma^2\rho(q^0, q^1, -q^2)\gamma^2)_{\alpha\beta}\theta(q^0)\,, \end{aligned} \tag{A.7}$$

with

$$\rho_{\alpha\beta}(q) = \sum_n \delta(q-p_n)\langle 0|\psi_\alpha(0)|n\rangle\langle n|\bar{\psi}_\beta(0)|0\rangle\,. \tag{A.8}$$

The sum in (A.8) is over positive energy states only. To derive (A.7) we have used PCT invariance (see [17] for a derivation in $d=4$). Because of Lorentz invariance

$$\rho_{\alpha\beta}(q)=\rho_1(q^2)\not{q}_{\alpha\beta}+\rho_2(q^2)\delta_{\alpha\beta}\,.$$

Assuming parity invariance, and using (A.8) in the form

$$\rho_{\alpha\beta}(q)=\frac{1}{(2\pi)^3}\int \mathrm{d}^3x\,\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(0)|0\rangle\,\mathrm{e}^{iqx}\,,$$

one easily derives that

$$\rho_2(q^2)=0\,. \tag{A.9}$$

After a Fourier transformation to momentum space

$$S_{\alpha\beta}(p)=(2\pi)^2\int_0^\infty \mathrm{d}s\,\rho_1(s)\frac{\not{p}}{p^2-s+i\varepsilon}\,. \tag{A.10}$$

This should be compared to (2.2). Rewriting (2.2) in the form

$$S_{\alpha\beta}(p)=\int_0^\infty \mathrm{d}\omega\frac{1}{p^2-\omega^2+i\varepsilon}[\not{p}(\rho(\omega)+\rho(-\omega))+\omega(\rho(\omega)-\rho(-\omega))]\,, \tag{A.11}$$

the parity-symmetric form (A.10) is seen to correspond to an even spectral function $\rho(\omega)$. The condition (A.9) sets parity-violating matrix elements equal to zero. For example (2.9):

$$\langle 0|\psi_\alpha(0)\bar{\psi}_\alpha(0)|0\rangle=8\pi\int_0^\infty \mathrm{d}\omega\,\omega^2\rho_2(\omega^2)=0\,. \tag{A.12}$$

Using (A.10) and (A.11) one identifies

$$\tfrac{1}{2}(\rho(\omega)+\rho(-\omega))=(2\pi)^2\omega\rho_1(\omega^2)\,,\qquad \omega>0\,. \tag{A.13}$$

Moment integrals of $\rho(\omega)$ can then be related to matrix elements of fermion operators. The constant A, defined in (2.19), is given by

$$\int_0^\infty \mathrm{d}\omega\,\omega^2\rho(\omega)=\frac{1}{(4\pi)^2}\int \mathrm{d}^3x\,\mathrm{d}^3q\langle 0|(i\,\not{\partial}\psi(x))\bar{\psi}(0)|0\rangle\frac{\mathrm{e}^{iq\,x}}{q}\,. \tag{A.14}$$

For the photon propagator we introduce the spectral function

$$\bar{\rho}_{\mu\nu}(k)=\frac{1}{(2\pi)^3}\int \mathrm{d}^3x\,\langle 0|A_\mu(x)A_\nu(0)|0\rangle\,\mathrm{e}^{ik\,x}\,, \tag{A.15}$$

which, because of Lorentz invariance, can be written as

$$\bar{\rho}_{\mu\nu}(k)=\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\bar{\rho}_1(k^2)+\frac{k_\mu k_\nu}{k^2}\bar{\rho}_2(k^2)+i\varepsilon_{\mu\nu\lambda}k^\lambda\bar{\rho}_3(k^2)\,. \tag{A.16}$$

Using gauge invariance and invariance under parity transformations we find that $\bar{\rho}_2$ and $\bar{\rho}_3$ are given by

$$\bar{\rho}_2(q^2) = -\frac{2a}{(2\pi)^2}\delta(q^2),$$

$$\bar{\rho}_3(q^2) = 0, \tag{A.17}$$

while $\bar{\rho}_1$ is related to the spectral function $\bar{\rho}$ of (2.14) by

$$\bar{\rho}(\mu) = -2(2\pi)^2\mu\bar{\rho}_1(\mu^2). \tag{A.18}$$

This allows the following identification of moment integrals:

$$\frac{1}{\pi}\int_0^\infty d\mu\,\mu\bar{\rho}(\mu) = -\langle 0|A_\mu(0)A^\mu(0)|0\rangle, \tag{A.19}$$

$$\frac{2}{\pi}\int_0^\infty d\mu\,\mu^3\bar{\rho}(\mu) = -\langle 0|F_{\mu\nu}(0)F^{\mu\nu}(0)|0\rangle. \tag{A.20}$$

In the derivation of (A.20) boundary terms from partial integrations, which are not manifestly gauge invariant, have been neglected.

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